

A Note on Polarization Vectors in Quantum Electrodynamics

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Abstract

A photon of momentum k can have only two polarization states, not three. Equivalently, one can say that the magnetic vector potential A must be divergence free in the Coulomb gauge. These facts are normally taken into account in QED by introducing two polarization vectors $\varepsilon_\lambda(k)$ with $\lambda \in \{1, 2\}$, which are orthogonal to the wave-vector k . These vectors must be very discontinuous functions of k and, consequently, their Fourier transforms have bad decay properties. Since these vectors have no physical significance there must be a way to eliminate them and their bad decay properties from the theory. We propose such a way here.

1 Introduction

In quantum electrodynamics it is necessary to choose a gauge for the electromagnetic field, and we shall use the Coulomb gauge here for the reason that it is the only gauge in which one gets the correct electromagnetic field and electromagnetic energy by a minimization principle. This comes from the fact that the magnetostatic interaction energy of a current distribution can be found by minimizing $(8\pi)^{-1} \int B^2 - \int j \cdot \mathbf{A}$ with respect to the vector potential whereas the (positive) electrostatic interaction energy of a charge distribution ρ is not given by the minimum of $8\pi^{-1} \int |\nabla \phi|^2 - \int \phi \rho$. (For a discussion of this see, e.g., [1].) In the Coulomb gauge the electrostatic part of the interaction among particles is given directly by Coulomb's law $e_i e_j / |x_i - x_j|$. The curl-free part of the electric field is not a dynamical variable in this gauge.

The dynamical field, whose dynamics is 'quantized', is the magnetic field. The (ultraviolet cutoff) magnetic vector potential is customarily defined by

$$\mathbf{A}(x) = \frac{\sqrt{\hbar c}}{2\pi} \sum_{\lambda=1}^2 \int_{\mathbb{R}^3} \frac{\varepsilon_\lambda(k)}{\sqrt{|k|}} \hat{\chi}_\lambda(k) \left(a_\lambda(k) e^{ik \cdot x} + a_\lambda^*(k) e^{-ik \cdot x} \right) dk, \quad (1.1)$$

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where the function $\widehat{\chi}_\Lambda$ is a radial function in k space that vanishes outside the ball whose radius is the ultraviolet cutoff Λ . The creation and annihilation operators of photons of momentum k and polarization λ , $a_\lambda(k)$ and $a_\lambda^*(k)$, which act on Fock-space (over $L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$), satisfy the canonical commutation relations

$$[a_\lambda(k), a_\nu^*(q)] = \delta(k - q)\delta_{\lambda,\nu}, \quad [a_\lambda(k), a_\nu(q)] = 0, \quad \text{etc.} \quad (1.2)$$

The magnetic field is $\mathbf{B}(x) = \text{curl } \mathbf{A}(x)$. The vectors $\varepsilon_\lambda(k)$ are two orthonormal polarization vectors, which are perpendicular to k as well as to each other.

The field energy, H_f , sometimes called $d\Gamma(\omega)$, is given by

$$H_f = \hbar c \sum_{\lambda=1,2} \int_{\mathbb{R}^3} |k| a_\lambda^*(k) a_\lambda(k) dk. \quad (1.3)$$

There is no cutoff in H_f . The energy of a photon is $\hbar c|k|$.

The polarization vectors are necessarily discontinuous. They must be discontinuous on every sphere centered at the origin, in fact, for the well known reason that “one cannot comb the hair on a sphere”. One possible choice, but by no means the only one, is

$$\begin{aligned} \varepsilon_1(k) &= \frac{(k_2, -k_1, 0)}{\sqrt{k_1^2 + k_2^2}}, \\ \varepsilon_2(k) &= \frac{k}{|k|} \wedge \varepsilon_1(k). \end{aligned} \quad (1.4)$$

While polarization is physically meaningful and measurable, the polarization vectors and the corresponding operators $a_\lambda^\#(k)$ have no direct physical meaning. The polarization *vectors* merely form an arbitrarily chosen and hence *unobservable* basis for vectors perpendicular to k . It should be possible to define the theory without the unphysical operators $a_\lambda^\#(k)$.

Why does this matter? For an atom in the ground state Φ one expects that the photon density decays towards zero with increasing distance from the nucleus. Since all the relevant quantities of the radiation field are expressed in terms of operators that act on k space, it is convenient to establish this decay by finding a bound on

$$\sum_{\lambda=1}^2 \int \|\nabla_k a_\lambda(k) \Phi\|^2 dk, \quad (1.5)$$

noting that smoothness in k space translates into decay in configuration space. To calculate the above expression one needs to compute $\nabla_k \varepsilon_\lambda(k)$ whose singularity causes needless complications, as in [2, 3].

A related complication caused by the polarization vectors occurs if one wants to calculate the coupling function $h_\lambda^i(x)$ that measures the strength of the interaction of the electron with the photon field. This function is given by the Fourier transform of the k -dependent quantity appearing in (1.1), namely (with superscripts $i = 1, 2, 3$ denoting components)

$$h_\lambda^i(y) = \frac{1}{2\pi} \int \frac{\widehat{\chi}_\Lambda(k)}{\sqrt{|k|}} \varepsilon_\lambda^i(k) e^{-ik \cdot y} dk \quad (1.6)$$

The reason for the interest in $h_\lambda^i(y)$ is that in some problems, such as the verification of the binding condition for atoms [4] or the existence of the thermodynamic limit, it is necessary to localize the electromagnetic field in x -space. Thus, if we formally define

$$\check{a}_\lambda(x) = \frac{1}{(2\pi^3)} \int_{\mathbb{R}^3} a_\lambda(k) e^{ik \cdot x} dk \quad (1.7)$$

to be the Fourier transform of the operator $a_\lambda(k)$ then

$$A^i(x) = \sum_{\lambda=1}^2 \check{a}_\lambda(h_\lambda^i(x - \cdot) + \check{a}_\lambda^*(h_\lambda^i(x - \cdot)) . \quad (1.8)$$

(Formally, $\check{a}_\lambda(h_\lambda^i(x - \cdot) = \int \check{a}_\lambda(y) h_\lambda^i(x - y) dy$.) We want h_λ^i to have a rapid fall-off in order that $\check{a}_\lambda(y)$ be localized with y as close to x as possible.

Unfortunately, because of the discontinuity of $\varepsilon_\lambda^i(k)$ it is very difficult to decide what the fall-off of $h_\lambda^i(y)$ is. The decay, being slow in directions that are perpendicular to the direction of the singularity of $\varepsilon_\lambda^i(k)$, will be nonuniform. With a smooth cutoff function $\hat{\chi}_\Lambda(k)$ we can, by a suitable choice of the polarization vectors, make $h_\lambda^i(y)$ decay in such a way that $\int |x|^{2\gamma} |h_\lambda^i(y)|^2 dy$ is finite for all $\gamma < 1$. This weak form of fall-off is useful, but inconvenient to work with. Moreover, one can envision situations where the nonuniformity of the decay will cause problems.

In contrast, if we omit the $\varepsilon_\lambda^i(k)$ then the fall-off of the basic coupling function

$$h(y) = \frac{1}{2\pi} \int \frac{\hat{\chi}_\Lambda(k)}{\sqrt{|k|}} e^{-ik \cdot y} dk , \quad (1.9)$$

will be $|y|^{-5/2}$ as $|y| \rightarrow \infty$ if we choose the cutoff $\hat{\chi}_\Lambda(k)$ to be smooth. To see this we note that $|k|^{-1/2}$ is the Fourier transform of $|y|^{-5/2}$ in the sense of distributions [5, Theorem 5.9]. The Fourier transform of $\hat{\chi}_\Lambda$ is real analytic and decays faster than any inverse power of $|y|$. Hence, the convolution of χ_Λ (the Fourier transform of $\hat{\chi}_\Lambda$), with $|y|^{-5/2}$ decays like $|y|^{-5/2}$. As an aside we note that a sharp ultraviolet cutoff, $h(y)$ would decay only like $|y|^{-2}$, which turns out to be insufficient for a good localization of the photon states.

In an attempt to eliminate the polarization vectors from the formalism it was suggested in [4] that it would be better to start with a formalism that contains only “divergence-free” vector fields as the dynamical variables. In particular, the Fock space would be built over the L^2 -space of divergence-free vector fields instead of $L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$. While this approach removes the arbitrariness in the choice of polarizations it causes problems when one attempts to localize photons. If one multiplies a divergence-free vector field v by a smooth cutoff function j the result is not a divergence-free vector field in general. One possible localization procedure is to write $v = \text{curl } w$ and then use the field $jv + \nabla j \wedge w$. Again, this is fairly tedious.

Since then, we have found an easier way and that is the subject of this paper.

2 The Third Photon Mode

Our proposal is really very simple. Let us introduce three photon degrees of freedom for each $k \in \mathbb{R}^3$ namely,

$$\mathbf{a}(k) = \{a^1(k), a^2(k), a^3(k)\}, \quad (2.1)$$

with the a^i satisfying canonical commutation relations $[a^i(k), a^j(q)] = 0$ and $[a^i(k), a^{j*}(q)] = \delta_{i,j}\delta(k-q)$. Another way to say this is that we use the Fock space over $L^2(\mathbb{R}^3) \otimes \mathbb{C}^3$. The field energy involves all three modes:

$$H_f = \hbar c \sum_{j=1}^3 \int_{\mathbb{R}^3} |k| a^{j*}(k) a^j(k) dk . \quad (2.2)$$

The vector potential is replaced by

$$\mathbf{A}(x) = \frac{1}{2\pi} \int_{\mathbb{R}^3} \frac{1}{\sqrt{|k|}} \hat{\chi}_\Lambda(k) \frac{k}{|k|} \wedge \left(\mathbf{a}(k) e^{ik \cdot x} + \mathbf{a}^*(k) e^{-ik \cdot x} \right) dk , \quad (2.3)$$

and the analogue of (1.8) is the vector equation

$$\mathbf{A}(x) = i \operatorname{curl} \tilde{\mathbf{a}}(\tilde{h}(x - \cdot)) - i \operatorname{curl} \tilde{\mathbf{a}}^*(\tilde{h}(x - \cdot)) . \quad (2.4)$$

The function \tilde{h} is related to (1.9)

$$\tilde{h}(y) = \frac{1}{2\pi} \int \frac{\hat{\chi}_\Lambda(k)}{|k|^{3/2}} e^{-ik \cdot y} dk . \quad (2.5)$$

Note that

$$\left\{ \operatorname{curl} \tilde{\mathbf{a}}(\tilde{h}(x - \cdot)) \right\}^\ell = \sum_{j=1}^3 \left\{ \tilde{a}^j(\partial_i \tilde{h}(x - \cdot)) - \tilde{a}^i(\partial_j \tilde{h}(x - \cdot)) \right\} \varepsilon_{ij\ell} . \quad (2.6)$$

Following the argument about the fall-off of $h(y)$, we see that the fall-off of $\tilde{h}(y)$ is bounded by $|y|^{-3/2}$ and its derivatives are bounded by $|y|^{-5/2}$. Thus, the desired fall-off $|y|^{-5/2}$ is obtained, using (2.6), for the localization of $\mathbf{A}(x)$.

We have the following situation: The quantized field operators have been increased from two to three for each k -value. Nevertheless, the vector potential \mathbf{A} (which mediates the interaction of matter and radiation) still has the property that its k -Fourier transform is perpendicular to k . The field energy has been supplemented by an additional mode (which we might think of as 'dark energy' since it does not interact with matter and is, therefore, not detectable). For reasons outlined at the beginning of this paper these additional modes are not longitudinal photons. Those have been eliminated from the theory by choosing the Coulomb gauge.

What we shall show next is that this theory, in which the Hamiltonian describing matter and its interaction with radiation (relativistic or non-relativistic) is formally the same except for the extra invisible mode, gives the same physics as the old theory with the two polarization vectors.

Before continuing, let us note that there is an alternative to (2.3) that accomplishes the same thing. We can replace the vector $|k|^{-1} k \wedge \mathbf{a}(k)$ in (2.3), whose ℓ -component is $|k|^{-1} k^i a^j(k) \varepsilon_{ij\ell}$, by the vector whose ℓ -component is

$$\sum_{\lambda=1}^2 \sum_{j=1}^3 \varepsilon_\lambda^\ell(k) \varepsilon_\lambda^j(k) a^j(k) = \left\{ \sum_{\lambda=1}^2 \varepsilon_\lambda(k) \varepsilon_\lambda(k) \cdot \mathbf{a}(k) \right\}^\ell = a^\ell(k) - \frac{k^\ell}{|k|^2} k \cdot \mathbf{a}(k) . \quad (2.7)$$

The vector whose ℓ -component is displayed in (2.7) is the projection of \mathbf{a} onto the plane perpendicular to the vector k . The vector $|k|^{-1} k \wedge \mathbf{a}(k)$ is a vector that is perpendicular to both \mathbf{a} and k and whose norm is the same as the vector in (2.7).

The theories with (2.7) and with $|k|^{-1} k \wedge \mathbf{a}(k)$ are indistinguishable. We shall continue with (2.3).

3 Equivalence of the Two Theories

Let us start with the theory defined by the vector potential (2.3) and field energy H_f in (2.2). That is to say, the total Hamiltonian of matter plus radiation has the form

$$H_{\text{total}} = H_{\text{matter}}(\mathbf{A}) + H_f \quad (3.1)$$

where $H_{\text{matter}}(\mathbf{A})$ describes the matter (as particles, or as a quantized field, relativistic or non-relativistic). The important point is that it depends on the radiation field only through the quantized vector field \mathbf{A} in (2.3).

Instead of the three k -dependent operators \mathbf{a} we introduce

$$a_\lambda(k) = \varepsilon_\lambda(k) \cdot \mathbf{a}(k), \quad a_0(k) = \frac{k}{|k|} \cdot \mathbf{a}(k) \quad (3.2)$$

and check that (with $\nu = \lambda$ or $\nu = 0$)

$$[a_\nu(k), a_\mu(q)] = 0 \quad [a_\nu(k), a_\mu^*(q)] = \delta_{\nu,\mu} \delta(k - q) . \quad (3.3)$$

We also observe that \mathbf{A} can be written using only the two $a_\lambda(k)$ as in (1.1) and that the field energy (2.2) is

$$H_f = \hbar c \int_{\mathbb{R}^3} |k| \left\{ \sum_{\lambda=1}^2 a_\lambda^*(k) a_\lambda(k) + a_0^*(k) a_0(k) \right\} dk . \quad (3.4)$$

All of this is breathtakingly elementary to verify. The conclusion, however, is interesting. This ‘three-component’ model (3.1), which is simpler to deal with than the usual ‘two-component’ model because the polarization vectors are absent, has the property of merely describing the ‘two-component’ theory plus one totally independent scalar field a_0 whose time evolution is governed by the Hamiltonian

$$H_0 = \hbar c \int_{\mathbb{R}^3} |k| a_0^*(k) a_0(k) dk . \quad (3.5)$$

Thus, the (Heisenberg) time evolution of a_0 is simply $a_0(k, t) = a_0(k) e^{ic|k|t}$.

The eigenvalues of (3.1), on the other hand, are those of the original ‘two-component’ model plus the scalar field energy, whose eigenvalues (if the radiation field is enclosed in a finite box) are of the form

$$\hbar c \sum |k| n_k . \quad (3.6)$$

The n_k are nonnegative integers, of which only a finite number are positive. For the ground state we choose all $n_k = 0$. The energy in the scalar field mode is not observable.

If we are interested in the thermodynamic limit of ordinary matter coupled to the radiation field in a positive temperature Gibbs state we have to proceed as follows. First, we imagine the universe to be a huge box of volume \mathcal{V} , while the matter is confined to a much smaller box of volume V . We would have to imagine this, even if we stayed with the usual ‘two-component’ formulation. Then we would take the limit $\mathcal{V} \rightarrow \infty$, after subtracting the well known positive temperature free energy $F(T)$ of the field, whose asymptotic value was calculated by Planck in 1900 [6].

$$\frac{F(T)}{k_{\text{Boltzmann}} T} = -3 \frac{\mathcal{V}}{(2\pi)^3} \int_{\mathbb{R}^3} \log \left(1 - \exp \left[-\frac{\hbar c |k|}{k_{\text{Boltzmann}} T} \right] \right) dk \quad (3.7)$$

This subtraction is necessary in either theory, the only difference being the subtraction of the scalar field energy in our ‘three-component’ theory. Hence the factor 3 instead of 2 in (3.7). After this $\mathcal{V} \rightarrow \infty$ limit, one takes the usual $V \rightarrow \infty$ limit. In the end, the scalar field contributes nothing. Its role is only to contribute some simplification to a difficult calculation.

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